ANALYTIC NILPOTENT CENTERS AS LIMITS OF NONDEGENERATE CENTERS REVISITED

ISAAC A. GARCÍA[1], HÉCTOR GIACOMINI[2], JAUME GINÉ[1] AND JAUME LLIBRE[3]

Abstract. We prove that all the nilpotent centers of planar analytic differential systems are limit of centers with purely imaginary eigenvalues, and consequently the Poincaré–Liapunov method to detect centers with purely imaginary eigenvalues can be used to detect nilpotent centers.

1. Introduction and statement of the main result

Consider the analytic family

$$\dot{x} = y + F_1(x, y; \lambda), \quad \dot{x} = F_2(x, y; \lambda) \tag{1}$$

with parameters $\lambda \in \mathbb{R}^p$ and having a nilpotent singularity at the origin.

In the papers [8–10] is stated a theorem which is slightly modified each time in order to correct the previous version but surprisingly it has never been properly written. Anyway we want to emphasize that the ideas presented in [8] have merit and in our opinion they are new and useful for understanding the nilpotent center problem mainly due to the computational algorithm that is derived from it. Here we present the correct statement and the right proof. We will present one counterexample for showing that the previous version of the mentioned theorem does not work. Also we end with an example for showing the analysis of a nilpotent center problem on a family using the right method provided here.

Here a *non-degenerate center* is a center with purely imaginary eigenvalues. See for example the book [12] for a modern treatment of the non-degenerate center problem and [6] to see some relationships with the Darboux integrability theory. In this work we focus on the nilpotent center problem which has been studied by several authors [1, 2, 5, 11].

Given an analytic function f at a point p, we say that f has *order* k at p if the Taylor series of f at p starts with terms of degree k in x and y.

Theorem 1. *Suppose that the origin of system* (1) *with* $\lambda = \lambda^*$ *is a nilpotent center. Then there are two (non unique) functions* $P(x, y)$ *and* $Q(x, y)$ *analytic at the origin and of order at least two such that the* 1*-parameter family*

$$\dot{x} = y + F_1(x, y; \lambda^*) + \varepsilon P(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y; \lambda^*) + \varepsilon Q(x, y) \tag{2}$$

possesses a non-degenerate center at the origin for any $\varepsilon > 0$. *Also there is an analytic function* $f(x, y)$ *at the origin of order at least two such that*

$$(x - Q)\frac{\partial f}{\partial y} = P\left(1 + \frac{\partial f}{\partial x}\right). \tag{3}$$

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Theorem 1 is proved in section 2.

The rest of the paper is organized as follows. In section 3 we do some remarks related to Theorem 1, and in section 4 using Theorem 1 we extend the algorithm of Poincaré-Liapunov for characterizing the nilpotent centers. Finally in sections 5 and 6 we provide the mentioned counterexample and example, respectively.

2. Proof of Theorem 1

Following [5] (see also [11]), if the analytic system (1) with $\lambda = \lambda^*$ has a center at the origin, then it is analytically orbitally equivalent to a time-reversible system. More precisely there exists an analytic near identity change of variables $(x, y) \mapsto (u, v) = \Phi(x, y) = (x + f(x, y), y + g(x, y))$ and a time rescaling $t \mapsto \tau(u, v)$ with $d\tau/dt = U(u, v)$ such that $U(0, 0) = 1$ and the new differentiable system is invariant under the involution $(u, v, \tau) \mapsto (-u, v, -\tau)$. More precisely, in the new variables (u, v) the differential system (1) becomes

$$\dot{u} = (v + \hat{F}_1(u^2, v))U(u, v), \quad \dot{v} = (u\hat{F}_2(u^2, v))U(u, v)$$

and after the time recaling we get the time-reversible system

$$u' = v + \hat{F}_1(u^2, v), \quad v' = u\hat{F}_2(u^2, v),$$

where the prime denotes derivative with respect to τ. Clearly the origin $(u, v) = (0, 0)$ is a nilpotent center. We perturb the system introducing the real parameter $\varepsilon > 0$ modifying the linear part as

$$u' = v + \hat{F}_1(u^2, v), \quad v' = -\varepsilon u + u\hat{F}_2(u^2, v). \tag{4}$$

Now the origin is a non-degenerate reversible center for any $\varepsilon > 0$. We go back to the initial time variable t and we obtain that the differential system

$$\dot{u} = (v + \hat{F}_1(u^2, v))U(u, v), \quad \dot{v} = (-\varepsilon u + u\hat{F}_2(u^2, v))U(u, v) \tag{5}$$

also has a non-degenerate center at the origin for any $\varepsilon > 0$ because $U(0, 0) = 1$. Using the chain rule we have

$$\dot{u} = \dot{x} + \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}, \quad \dot{v} = \dot{y} + \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}$$

and inverting we get

$$\dot{x} = \frac{1}{\Delta(x, y)}\left[\left(1 + \frac{\partial g}{\partial y}\right)\dot{u} - \frac{\partial f}{\partial y}\dot{v}\right], \quad \dot{y} = \frac{1}{\Delta(x, y)}\left[\left(1 + \frac{\partial f}{\partial x}\right)\dot{v} - \frac{\partial g}{\partial x}\dot{u}\right], \tag{6}$$

where

$$\Delta(x, y) = 1 + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

From here it is easy to pull back (5) to the original variables (x, y) obtaining system (2) with

$$\begin{aligned} P(x, y) &= \frac{U \circ \Phi(x, y)}{\Delta(x, y)}(x + f(x, y))\frac{\partial f}{\partial y}, \\ Q(x, y) &= x - \frac{U \circ \Phi(x, y)}{\Delta(x, y)}(x + f(x, y))\left(1 + \frac{\partial f}{\partial x}\right). \end{aligned} \tag{7}$$

Therefore it is evident that $P(x, y)$ and $Q(x, y)$ are analytic and have order at least two at $(x, y) = (0, 0)$ and also that (3) holds. This completes the proof of Theorem 1.

3. Remarks

Remark 2. We emphasize that the expressions for the functions P and Q in the analogous to Theorem 1 given in the references [8–10] are wrong. More specifically in [8] it is stated that $P(0,y) = Q(0,y) \equiv 0$, and in [9] that P and Q only depend on f in the specific way

$$P(x,y) = (x + f(x,y))\frac{\partial f}{\partial y}, \quad Q(x,y) = -(x + f(x,y))\frac{\partial f}{\partial x} - f(x,y). \tag{8}$$

The error in the proof was the assumption that nilpotent centers are conjugated to a time-reversible system, hence without performing the time rescaling $t \mapsto \tau$. Later on in [10] it is taking into account that actually nilpotent centers are orbitally conjugated to time-reversible systems but in the proof also appear mistakes yielding again that P and Q only depend on f as in (8). Our proof shows that P and Q depend on f, g and U. In particular the computations involved with the method associated to Theorem 1 to detect nilpotent centers become harder since now we have more freedom in choosing P and Q. Anyway all the center conditions founded in all the examples studied in [8, 10] are correct either because we have checked them using Theorem 1, or because they have been studied by other authors.

Remark 3. Instead of perturbing as in (4) we can perturb in the following form

$$u' = v + \hat{F}_1(u^2, v) - \varepsilon \hat{G}_1(u^2, v), \quad v' = -\varepsilon u + u\hat{F}_2(u^2, v) - \varepsilon u \hat{G}_2(u^2, v),$$

with any pair of analytic functions $\hat{G}_1(u^2, v)$ and $\hat{G}_2(u^2, v)$ starting in at least second and first order respectively. In this case we obtain system (2) with

$$\begin{aligned} P(x,y) &= \frac{U \circ \Phi}{\Delta}\left[(1 + \hat{G}_2 \circ \Phi)(x + f)\frac{\partial f}{\partial y} - \left(1 + \frac{\partial g}{\partial y}\right)(\hat{G}_1 \circ \Phi)\right], \\ Q(x,y) &= x - \frac{U \circ \Phi}{\Delta}\left[(1 + \hat{G}_2 \circ \Phi)(x + f)\left(1 + \frac{\partial f}{\partial x}\right) - (\hat{G}_1 \circ \Phi)\frac{\partial g}{\partial x}\right]. \end{aligned}$$

This proves that P and Q are not unique due to the arbitrariness of the functions $\hat{G}_1$ and $\hat{G}_2$. In the particular case when $\hat{G}_1 = \hat{G}_2 \equiv 0$ we recover the former analysis in the proof of Theorem 1.

It is worth to emphasize that, in general, we cannot choose $\hat{G}_2$ satisfying the functional equation $(U \circ \Phi)/\Delta = 1 + \hat{G}_2 \circ \Phi$. The reason is that if it was possible, then adding $\hat{G}_1 \equiv 0$ we recover the expressions (8). But we know (see the section where we analyze system (10)) that this is not possible.

Remark 4. Using (3) we see that x must factor out the homogeneous polynomial of minimal degree in the Taylor expansion of P at the origin. This means that the function P has a Taylor expansion of the form $P(x,y) = \sum_{i+j \geq 2} p_{ij} x^i y^j$ with the coefficient $p_{02} = 0$.

Remark 5. Clearly Theorem 1 provides necessary center conditions but only when it is applied to a family (1) having a monodromic nilpotent singularity at the origin. For example if the origin of system (1) with $\lambda = \lambda^\dagger$ is not monodromic but it is time-reversible with respect to the involution $(x, y, t) \mapsto (-x, y, -t)$, then the perturbation $\dot{x} = y + F_1(x, y; \lambda^\dagger)$, $\dot{y} = -\varepsilon x + F_2(x, y; \lambda^\dagger)$ has a time-reversible non-degenerate center at the origin for any $\varepsilon > 0$ showing that (2) holds with $P = Q \equiv 0$.

In summary, since we will only apply Theorem 1 to monodromic nilpotent families (1) we recall the following theorem of Andreev that characterizes that property.

Theorem 6 ([3]). *For an analytic system of the form* (1) *with* $\lambda = \hat{\lambda}$ *and having an isolated singularity at the origin let* $y = F(x)$ *be the unique solution of* $y + F_1(x, y; \hat{\lambda}) = 0$ *such that* $F(0) = F'(0) = 0$, *and let*

$$f(x) = F_1(x, F(x); \hat{\lambda}) \qquad \text{and} \qquad \xi(x) = (\partial F_1/\partial x + \partial F_2/\partial y)(x, F(x)).$$

Let $a \neq 0$ *and* $\alpha \geq 2$ *be such that* $f(x) = ax^\alpha + \cdots$. *When* ξ *is not identically zero let* $b \neq 0$ *and* $\beta \geq 1$ *be such that* $\xi(x) = bx^\beta + \cdots$. *Then the origin of* (1) *with* $\lambda = \hat{\lambda}$ *is monodromic if and only if* $\alpha = 2n - 1$ *is an odd integer,* $a < 0$, *and one of the following conditions holds:*
(i) $\xi(x) \equiv 0$,
(ii) $\beta \geq n$,
(iii) $\beta = n - 1$ *and* $b^2 + 4an < 0$.

4. The Poincaré-Liapunov algorithm

Let (1) be a family of differential systems having a monodromic nilpotent singularity at the origin. Then from Theorem 1 we derive an algorithm to determine necessary conditions on the parameters of the family for having a nilpotent center at the origin. Of course this algorithm is just the well known Poincaré-Liapunov method applied to the larger perturbed family (2) as it is explained in [8]. More specifically, since (2) has an analytic first integral for any $\varepsilon > 0$ there are *focus quantities* $\eta_i(\lambda, \varepsilon)$ and a formal series $H(x, y; \lambda, \varepsilon) = \varepsilon x^2 + y^2 + \cdots$ such that

$$\mathcal{X}_\varepsilon(H) = \sum_{i \geq 2} \eta_i(\lambda, \varepsilon)(x^2 + y^2)^i, \tag{9}$$

where $\mathcal{X}_\varepsilon = [y + F_1(x, y; \lambda) + \varepsilon P(x, y)]\partial_x + [-\varepsilon x + F_2(x, y; \lambda) + \varepsilon Q(x, y)]\partial_y$ is the vector field associated to family (2) with arbitrary λ. Then system (2) with $\lambda = \lambda^*$ has a center at the origin for any $\varepsilon > 0$ if and only if $\eta_i(\lambda^*, \varepsilon) \equiv 0$ for all $i \geq 2$.

In practice and using a computer algebra system such as Mathematica we find the first terms of a formal series $H(x, y; \lambda, \varepsilon) = \varepsilon x^2 + y^2 + \sum_{j+k \geq 3} h_{jk}(\lambda, \varepsilon) x^j y^k$ satisfying (9). Equating the terms of homogeneous degree d we get the expressions of $h_{jk}(\lambda, \varepsilon)$ for $j + k = d$ when d is odd, and the functions $h_{jk}(\lambda, \varepsilon)$ for $j + k = d$ and $j \neq 0$, together with the focal value $\eta_{d/2}(\lambda, \varepsilon)$ when d is even. If d is even there appears an arbitrariness because you can select to solve, besides $\eta_{d/2}(\lambda, \varepsilon)$, a different set of d variables $h_{jk}(\lambda, \varepsilon)$ with $j + k = d$ and not necessarily the set with $j \neq 0$.

Remark 7. Although it is clear that all the η_j and H also depend on the coefficients of the analytic perturbation field (P, Q) we simplify the notation and we only write its dependence on (λ, ε). The computation of the quantities $\eta_j(\lambda, \varepsilon)$ in the described algorithm with $\varepsilon \neq 0$ is completely standard since $\mathcal{X}_\varepsilon$ has a nondegenerate center at the origin. The detailed steps of the process can be found in textbooks such as [12]. From there you can see that neither $\eta_j(\lambda, \varepsilon)$ nor the Liapunov function $H(x, y; \lambda, \varepsilon)$ are determined with uniqueness. What is unique is, for a fixed $\varepsilon = \varepsilon^* \neq 0$, the polynomial ideal $\mathcal{B}$ generated by all the polynomials $\{\eta_j(\lambda, \varepsilon^*)\}_{j \in \mathbb{N}}$, hence the real variety $\mathbf{V}_\mathbb{R}(\mathcal{B})$ associated to $\mathcal{B}$. However we do not work with a fixed ε, on the contrary we compute the expressions of $\eta_j(\lambda, \varepsilon)$ and $H(x, y; \lambda, \varepsilon)$ for any ε and, clearly, we can have some $(j,k) \in \mathbb{N}^2$ such that $\lim_{\varepsilon \to 0} h_{jk}(\lambda, \varepsilon) = \infty$. Notice that the particular case of having $H(x, y; \lambda, 0)$ and $\eta_j(\lambda, 0)$ well defined for all $j \in \mathbb{N}$, the vector field $\mathcal{X}_0$ with parameters $\lambda = \lambda^*$ satisfying $\eta_j(\lambda^*, 0) = 0$ for any $j \in \mathbb{N}$ has a formal first integral $H(x, y; \lambda^*, 0)$ and, in consequence, there is an analytic first integral near the nilpotent singularity at the origin of $\mathcal{X}_0$ for the specific choice of parameters $\lambda = \lambda^*$.

On the other hand, by construction, it is easy to see that the $\eta_j(\lambda, \varepsilon)$ are rational functions of ε and polynomial functions of λ. Of course η_j also depend on a finite number parameters given by the coefficients $\mu \in \mathbb{R}^s$ of the polynomial cut $(P, Q) = (\sum_{i=2}^{j-1} P_i(x, y; \mu), \sum_{i=2}^{j-1} Q_i(x, y; \mu))$ of the perturbation field (P, Q). Here (P_i, Q_i) is a homogeneous polynomial vector field of degree i. In summary, one obtain that for any $j \in \mathbb{N}$,

$$\eta_j(\lambda, \mu, \varepsilon) = \frac{A_j(\lambda, \mu, \varepsilon)}{B_j(\varepsilon)}$$

with $B_j \in \mathbb{R}[\varepsilon]$ and $A_j \in \mathbb{R}[\lambda, \mu, \varepsilon]$. In particular, $A_j(\lambda, \mu, \varepsilon) = \sum_{i=0}^{d_j} a_{i,j}(\lambda, \mu)\varepsilon^i$ for some $a_{i,j} \in \mathbb{R}[\lambda, \mu]$ and, consequently, $\eta_j = 0$ for all $\varepsilon \neq 0$ if and only if $a_{i,j}(\lambda, \mu) = 0$ for all $i = 0, 1, \ldots, d_j$. The nilpotent center conditions for the polynomial vector field $\mathcal{X}_0$ appear when some $a_{i,j}$ only depends on λ and not of μ. Of course, if the monodromic nilpotent family $\mathcal{X}_0$ contains both centers and foci for different parameter values λ then there must exist some of the above polynomials $a_{i,j}$ only depending on λ. Let $V_{i,j}(\lambda) = a_{i,j}(\lambda) \in \mathbb{R}[\lambda]$ for all $(i, j) \in \mathbb{N}^2$ such that $a_{i,j}$ does not depend on μ. We define the polynomial ideal $\mathcal{I}$ generated by all the polynomials $V_{i,j}$ in the ring $\mathbb{R}[\lambda]$.

We define the *nilpotent center set* $\mathcal{C} \subset \mathbb{R}^p$ as

$$\mathcal{C} = \{\lambda^* \in \mathbb{R}^p : \mathcal{X}_0 \text{ with } \lambda = \lambda^* \text{ has a center at the origin. } \}.$$

From Theorem 1 we know that $\mathcal{C} \subseteq \mathbf{V}_{\mathbb{R}}(\mathcal{I})$, the associated real variety to $\mathcal{I}$. Despite all the arbitrariness shown in all the process what remains invariant is just the ideal $\mathcal{I}$ and also the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{I})$. Of course, if $\lambda^\dagger \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$, we still do not know if $\lambda^\dagger$ lies in $\mathcal{C}$ or not. To check whether $\lambda^\dagger \in \mathcal{C}$ we need to use ad hoc tools based mainly on integrability theory or symmetry properties. For example, if the nilpotent singularity is monodromic, then sufficient conditions that guarantee that it is a center are either the existence of an analytic first integral in a neighborhood of it or the existence of a reversal symmetry. We want to emphasize here that there are analytic nilpotent centers which are not analytically integrable nor reversible, hence other methods must be used such as the orbital reversibility.

5. A Counterexample

We will show that the method proposed in [9, 10] fails to detect all nilpotent centers in a nilpotent family. We shall take a family of differential systems having a monodromic nilpotent singularity at the origin and we will prove that if we use the old wrong version of Theorem 1 with P and Q given by (8) with an arbitrary analytic function f having at least second order, then we obtain more restrictions than the necessary and sufficient ones. This will show that the method of [9, 10] does not work.

We consider the following polynomial differential systems of degree 7:

$$
\begin{aligned}
\dot{x} = {} & y + F_1(x, y; \lambda) \\
= {} & y + Ax^6y + Bx^5y^2 + Cx^4y^3 + Dx^3y^4 + Fx^2y^5 + Gxy^6 + Hy^7, \\
\dot{y} = {} & F_2(x, y; \lambda) \\
= {} & -x^7 + Vx^6y + Kx^5y^2 + Lx^4y^3 + Mx^3y^4 + Nx^2y^5 + Pxy^6 + Qy^7,
\end{aligned} \tag{10}
$$

with parameters $\lambda = (A, B, C, D, F, G, H, V, K, L, M, N, P, Q) \in \mathbb{R}^{14}$. In [4] it is proved that the origin is a center if and only if one of the following conditions is satisfied:

(i) Hamiltonian: $V = 3A + K = 5B + 3L = C + M = 3D + 5N = F + 3P = G + 7Q = 0$;
(ii) Time-reversible: $V = B = D = G = L = N = Q = 0$;
(iii) $V = 3A+K = 5B+3L = 3D+5N = F+3P-2A(C+M) = 5(G+7Q)+4L(C+M) = 2KL+5N = 25P+4L^2-25A(2A^2+M) = 5Q+L(2A^2+M) = 0$.

We will see that there are centers in (iii) which are not detected by the method of [9,10] when it is applied with $P(x, y)$ and $Q(x, y)$ given by the misleading expression (8). Performing the Poincaré-Liapunov algorithm on family (10) with $P(x, y)$ and $Q(x, y)$ given by (8) for some $f(x, y; \lambda) = \sum_{j+k\geq 2} f_{jk}(\lambda)x^jy^k$ we obtain $\eta_2(\lambda, \varepsilon) = \eta_3(\lambda, \varepsilon) \equiv 0$ but

$$
\eta_4(\lambda, \varepsilon) = \frac{2\varepsilon[5V + (5B + 3L)\varepsilon + (3D + 5N)\varepsilon^2 + 5(G + 7Q)\varepsilon^3]}{35 + 20\varepsilon + 18\varepsilon^2 + 20\varepsilon^3 + 35\varepsilon^4}.
$$

Imposing $\eta_4(\lambda, \varepsilon) \equiv 0$ for any $\varepsilon > 0$ produces the parameter conditions $V = 5B + 3L = 3D + 5N = G + 7Q = 0$. Clearly the last condition, $G + 7Q = 0$, is a wrong center condition as one can see from the conditions (iii) of one component of the center variety.

6. Example

We give the right proof of Proposition 6 of [8].

Proposition 8. *System $\dot{x} = y + x^2 + k_2xy$, $\dot{y} = -x^3 + k_1x^2$ has a nilpotent center at the origin if and only if $k_1 = k_2 = 0$.*

Proof. Let $\mathcal{X}_\varepsilon$ be the vector field associated to family (2) with $F_1(x, y; \lambda) = x^2 + k_2xy$, $F_2(x, y; \lambda) = -x^3 + k_1x^2$ and parameters $\lambda = (k_1, k_2) \in \mathbb{R}^2$. We define $P(x, y) = \sum_{i+j\geq 2} p_{ij}x^iy^j$ and $Q(x, y) = \sum_{i+j\geq 2} q_{ij}x^iy^j$ two analytic functions of order greater or equal than 2 at the origin with coefficient $p_{02} = 0$. We start the Poincaré-Liapunov algorithm for this family obtaining

$$
\eta_2(\lambda, \varepsilon) = \frac{2}{3 + 2\varepsilon + 3\varepsilon^2}(2k_1 + O(\varepsilon)).
$$

The vanishing of η_2 for any $\varepsilon > 0$ implies the parameter condition $k_1 = 0$. In the next step we compute η_2 and we choose parameters such that $\eta_2 \equiv 0$. Here we have freedom to select several parameters and we have choose to solve for q_{20}, q_{21} and p_{12}. These parameters are uniquely determined in terms of other parameters. Next we obtain

$$
\eta_3(\lambda, \varepsilon) = \frac{-1}{6(1 + \varepsilon)(5 - 2\varepsilon + 5\varepsilon^2)}(36k_2 + O(\varepsilon)).
$$

From here we deduce the center condition $k_2 = 0$. Of course this condition is independent of the former choice of vanishing η_2.

To see that actually system $\dot{x} = y + x^2$, $\dot{y} = -x^3$ has a nilpotent center at the origin it is sufficient to note that the origin is monodromic (just apply to it Andreev's Theorem 6) and that the system is time-reversible with respect to the involution $(x, y, t) \mapsto (-x, y, -t)$. It is worth to emphasize that system $\dot{x} = y + x^2$, $\dot{y} = -x^3$ has no formal first integral, see [7]. However our method also detect this type of nonintegrable centers. □

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[1] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Catalonia, Spain

E-mail address: garcia@matematica.udl.cat, gine@matematica.udl.cat

[2] Laboratoire de Mathématique et Physique Théorique, CNRS (UMR 7350), Faculté des Sciences et Techniques, Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail address: Hector.Giacomini@lmpt.univ-tours.fr

[3] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat